5-5 Completing the Square

Main Ideas

- Solve quadratic equations by using the Square Root Property.
- Solve quadratic equations by completing the square.

New Vocabulary

completing the square

GET READY for the Lesson

Under a yellow caution flag, race car drivers slow to a speed of 60 miles per hour. When the green flag is waved, the drivers can increase their speed.

Suppose the driver of one car is 500 feet from the finish line. If the driver accelerates at a constant rate of 8 feet per second squared, the equation $t^2 + 22t + 121 = 246$ represents the time t it takes the driver to reach this line. To solve this equation, you can use the Square Root Property.

Square Root Property You have solved equations like $x^2 - 25 = 0$ by factoring. You can also use the Square Root Property to solve such an equation. This method is useful with equations like the one above that describes the race car's speed. In this case, the quadratic equation contains a perfect square trinomial set equal to a constant.

EXAMPLE 1 Equation with Rational Roots

Solve $x^2 + 10x + 25 = 49$ by using the Square Root Property.

$x^2 + 10x + 25 = 49$	Original equation
$(x + 5)^2 = 49$	Factor the perfect square trinomial.
$x + 5 = \pm\sqrt{49}$	Square Root Property
$x + 5 = \pm 7$	$\sqrt{49} = 7$
$x = -5 \pm 7$	Add -5 to each side.
$x = -5 + 7$ or $x = -5 - 7$	Write as two equations.
$x = 2$ $\quad$ $x = -12$	Solve each equation.

The solution set is $\{2, -12\}$. You can check this result by using factoring to solve the original equation.

CHECK Your Progress

Solve each equation by using the Square Root Property.

1A. $x^2 - 12x + 36 = 25$ $\qquad$ **1B.** $x^2 - 16x + 64 = 49$

Roots that are irrational numbers may be written as exact answers in radical form or as *approximate* answers in decimal form when a calculator is used.

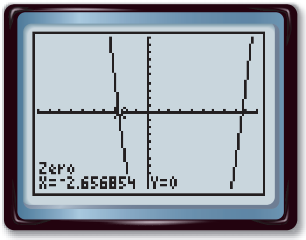
EXAMPLE 2 Equation with Irrational Roots

Solve $x^2 - 6x + 9 = 32$ by using the Square Root Property.

$x^2 - 6x + 9 = 32$	Original equation
$(x - 3)^2 = 32$	Factor the perfect square trinomial.
$x - 3 = \pm\sqrt{32}$	Square Root Property
$x = 3 \pm 4\sqrt{2}$	Add 3 to each side; $-\sqrt{32} = 4\sqrt{2}$
$x = 3 + 4\sqrt{2}$ or $x = 3 - 4\sqrt{2}$	Write as two equations.
$x \approx 8.7$ $\quad x \approx -2.7$	Use a calculator.

The exact solutions of this equation are $3 - 4\sqrt{2}$ and $3 + 4\sqrt{2}$. The approximate solutions are -2.7 and 8.7. Check these results by finding and graphing the related quadratic function.

$x^2 - 6x + 9 = 32$	Original equation
$x^2 - 6x - 23 = 0$	Subtract 32 from each side.
$y = x^2 - 6x - 23$	Related quadratic function



CHECK Use the ZERO function of a graphing calculator. The approximate zeros of the related function are -2.7 and 8.7.

Study Tip

Plus or Minus

When using the Square Root Property, remember to put a $\pm$ sign before the radical.

CHECK Your Progress

Solve each equation by using the Square Root Property.

2A. $x^2 + 8x + 16 = 20$ **2B.** $x^2 - 6x + 9 = 32$

Complete the Square The Square Root Property can only be used to solve quadratic equations when the quadratic expression is a perfect square. However, few quadratic expressions are perfect squares. To make a quadratic expression a perfect square, a method called **completing the square** may be used.

In a perfect square trinomial, there is a relationship between the coefficient of the linear term and the constant term. Consider the following pattern.

$$(x + 7)^2 = x^2 + 2(7)x + 7^2 \quad \text{Square of a sum pattern}$$

$$= x^2 + 14x + 49 \quad \text{Simplify.}$$

$$\left(\frac{14}{2}\right)^2 \rightarrow 7^2 \quad \text{Notice that 49 is } 7^2 \text{ and 7 is one half of 14.}$$

Use this pattern of coefficients to complete the square of a quadratic expression.

KEY CONCEPT — Completing the Square

Words To complete the square for any quadratic expression of the form $x^2 + bx$, follow the steps below.

Step 1 Find one half of b, the coefficient of x.

Step 2 Square the result in Step 1.

Step 3 Add the result of Step 2 to $x^2 + bx$.

Symbols $x^2 + bx + \left(\frac{b}{2}\right)^2 = x + \left(\frac{b}{2}\right)^2$

EXAMPLE Complete the Square

3 **Find the value of c that makes $x^2 + 12x + c$ a perfect square. Then write the trinomial as a perfect square.**

Step 1 Find one half of 12. $\frac{12}{2} = 6$

Step 2 Square the result of Step 1. $6^2 = 36$

Step 3 Add the result of Step 2 to $x^2 + 12x$. $x^2 + 12x + 36$

The trinomial $x^2 + 12x + 36$ can be written as $(x + 6)^2$.

CHECK Your Progress

3. Find the value of c that makes $x^2 - 14x + c$ a perfect square. Then write the trinomial as a perfect square.

Concepts in Motion
Animation
algebra2.com

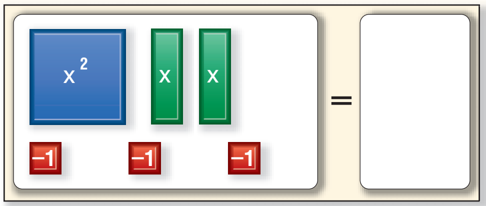
You can solve any quadratic equation by completing the square. Because you are solving an equation, add the value you use to complete the square to each side.

ALGEBRA LAB

Completing the Square

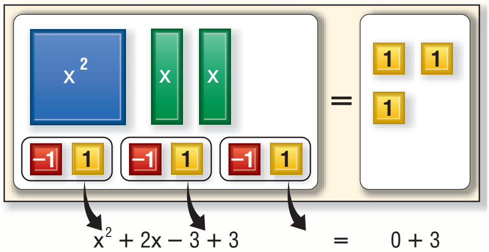
Use algebra tiles to complete the square for the equation $x^2 + 2x - 3 = 0$.

Step 1 Represent $x^2 + 2x - 3 = 0$ on an equation mat.



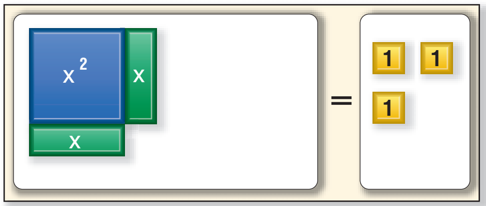
$x^2 + 2x - 3 \quad = \quad 0$

Step 2 Add 3 to each side of the mat. Remove the zero pairs.



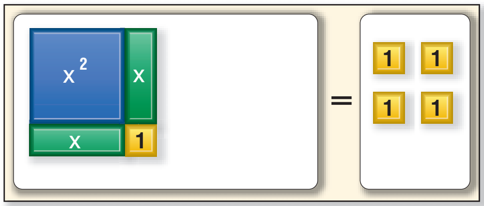
$x^2 + 2x - 3 + 3 \quad = \quad 0 + 3$

Step 3 Begin to arrange the x^2- and x-tiles into a square.



$x^2 + 2x \quad = \quad 3$

Step 4 To complete the square, add 1 yellow 1-tile to each side. The completed equation is $x^2 + 2x + 1 = 4$ or $(x + 1)^2 = 4$.



$x^2 + 2x + 1 \quad = \quad 3 + 1$

MODEL

Use algebra tiles to complete the square for each equation.

1. $x^2 + 2x - 4 = 0$

2. $x^2 + 4x + 1 = 0$

3. $x^2 - 6x = -5$

4. $x^2 - 2x = -1$

EXAMPLE Solve an Equation by Completing the Square

4 **Solve $x^2 + 8x - 20 = 0$ by completing the square.**

$x^2 + 8x - 20 = 0$	Notice that $x^2 + 8x - 20$ is not a perfect square.
$x^2 + 8x = 20$	Rewrite so the left side is of the form $x^2 + bx$.
$x^2 + 8x + 16 = 20 + 16$	Since $\left(\frac{8}{2}\right)^2 = 16$, add 16 to each side.
$(x + 4)^2 = 36$	Write the left side as a perfect square by factoring.
$x + 4 = \pm 6$	Square Root Property
$x = -4 \pm 6$	Add -4 to each side.
$x = -4 + 6$ or $x = -4 - 6$	Write as two equations.
$x = 2$ $\quad$ $x = -10$	The solution set is $\{-10, 2\}$.

You can check this result by using factoring to solve the original equation.

Study Tip

Common Misconception

When solving equations by completing the square, don't forget to add $\left(\frac{b}{2}\right)^2$ to *each* side of the equation.

CHECK Your Progress

Solve each equation by completing the square.

4A. $x^2 - 10x + 24 = 0$ **4B.** $x^2 + 10x + 9 = 0$

When the coefficient of the quadratic term is not 1, you must divide the equation by that coefficient before completing the square.

EXAMPLE Equation with $a \neq 1$

5 **Solve $2x^2 - 5x + 3 = 0$ by completing the square.**

$2x^2 - 5x + 3 = 0$	Notice that $2x^2 - 5x + 3$ is not a perfect square.
$x^2 - \frac{5}{2}x + \frac{3}{2} = 0$	Divide by the coefficient of the quadratic term, 2.
$x^2 - \frac{5}{2}x = -\frac{3}{2}$	Subtract $\frac{3}{2}$ from each side.
$x^2 - \frac{5}{2}x + \frac{25}{16} = -\frac{3}{2} + \frac{25}{16}$	Since $\left(-\frac{5}{2} \div 2\right)^2 = \frac{25}{16}$, add $\frac{25}{16}$ to each side.
$\left(x - \frac{5}{4}\right)^2 = \frac{1}{16}$	Write the left side as a perfect square by factoring. Simplify the right side.
$x - \frac{5}{4} = \pm \frac{1}{4}$	Square Root Property
$x = \frac{5}{4} \pm \frac{1}{4}$	Add $\frac{5}{4}$ to each side.
$x = \frac{5}{4} + \frac{1}{4}$ or $x = \frac{5}{4} - \frac{1}{4}$	Write as two equations.
$x = \frac{3}{2}$ $\quad$ $x = 1$	The solution set is $\left\{1, \frac{3}{2}\right\}$.

Study Tip

Mental Math

Use mental math to find a number to add to each side to complete the square.

$\left(-\frac{5}{2} \div 2\right)^2 = \frac{25}{16}$

CHECK Your Progress

Solve each equation by completing the square.

5A. $3x^2 + 10x - 8 = 0$ **5B.** $3x^2 - 14x + 16 = 0$

Not all solutions of quadratic equations are real numbers. In some cases, the solutions are complex numbers of the form $a + bi$, where $b \neq 0$.

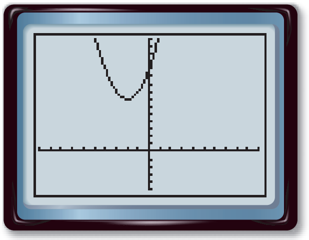
EXAMPLE 6 Equation with Complex Solutions

Solve $x^2 + 4x + 11 = 0$ by completing the square.

$x^2 + 4x + 11 = 0$	Notice that $x^2 + 4x + 11$ is not a perfect square.
$x^2 + 4x = -11$	Rewrite so the left side is of the form $x^2 + bx$.
$x^2 + 4x + 4 = -11 + 4$	Since $\left(\frac{4}{2}\right)^2 = 4$, add 4 to each side.
$(x + 2)^2 = -7$	Write the left side as a perfect square by factoring.
$x + 2 = \pm\sqrt{-7}$	Square Root Property
$x + 2 = \pm i\sqrt{7}$	$\sqrt{-1} = i$
$x = -2 \pm i\sqrt{7}$	Subtract 2 from each side.

The solution set is $\{-2 + i\sqrt{7}, -2 - i\sqrt{7}\}$. Notice that these are imaginary solutions.

CHECK A graph of the related function shows that the equation has no real solutions since the graph has no x-intercepts. Imaginary solutions must be checked algebraically by substituting them in the original equation.



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CHECK Your Progress

Solve each equation by completing the square.

6A. $x^2 + 2x + 2 = 0$ **6B.** $x^2 - 6x + 25 = 0$

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CHECK Your Understanding

Examples 1 and 2 (pp. 268–269)

Solve each equation by using the Square Root Property.

1. $x^2 + 14x + 49 = 9$
2. $x^2 - 12x + 36 = 25$
3. $x^2 + 16x + 64 = 7$
4. $9x^2 - 24x + 16 = 2$

Example 2 (p. 269)

ASTRONOMY For Exercises 5–7, use the following information.

The height h of an object t seconds after it is dropped is given by $h = -\frac{1}{2}gt^2 + h_0$, where h_0 is the initial height and g is the acceleration due to gravity. The acceleration due to gravity near Earth's surface is 9.8 m/s^2, while on Jupiter it is 23.1 m/s^2. Suppose an object is dropped from an initial height of 100 meters from the surface of each planet.

5. On which planet should the object reach the ground first?
6. Find the time it takes for the object to reach the ground on each planet to the nearest tenth of a second.
7. Do the times to reach the ground seem reasonable? Explain.

Example 3 (p. 270)

Find the value of c that makes each trinomial a perfect square. Then write the trinomial as a perfect square.

8. $x^2 - 12x + c$

9. $x^2 - 3x + c$

Examples 4–6 (pp. 271–272)

Solve each equation by completing the square.

10. $x^2 + 3x - 18 = 0$

11. $x^2 - 8x + 11 = 0$

12. $2x^2 - 3x - 3 = 0$

13. $3x^2 + 12x - 18 = 0$

14. $x^2 + 2x + 6 = 0$

15. $x^2 - 6x + 12 = 0$

Exercises

HOMEWORK HELP

For Exercises	See Examples
16–19, 40, 41	1
20–23	2
24–27	3
28–31	4
32–35	5
36–39	6

Solve each equation by using the Square Root Property.

16. $x^2 + 4x + 4 = 25$

17. $x^2 - 10x + 25 = 49$

18. $x^2 - 9x + \frac{81}{4} = \frac{1}{4}$

19. $x^2 + 7x + \frac{49}{4} = 4$

20. $x^2 + 8x + 16 = 7$

21. $x^2 - 6x + 9 = 8$

22. $x^2 + 12x + 36 = 5$

23. $x^2 - 3x + \frac{9}{4} = 6$

Find the value of c that makes each trinomial a perfect square. Then write the trinomial as a perfect square.

24. $x^2 + 16x + c$

25. $x^2 - 18x + c$

26. $x^2 - 15x + c$

27. $x^2 + 7x + c$

Solve each equation by completing the square.

28. $x^2 - 8x + 15 = 0$

29. $x^2 + 2x - 120 = 0$

30. $x^2 + 2x - 6 = 0$

31. $x^2 - 4x + 1 = 0$

32. $2x^2 + 3x - 5 = 0$

33. $2x^2 - 3x + 1 = 0$

34. $2x^2 + 7x + 6 = 0$

35. $9x^2 - 6x - 4 = 0$

36. $x^2 - 4x + 5 = 0$

37. $x^2 + 6x + 13 = 0$

38. $x^2 - 10x + 28 = 0$

39. $x^2 + 8x + 9 = -9$

40. **MOVIE SCREENS** The area A in square feet of a projected picture on a movie screen is given by $A = 0.16d^2$, where d is the distance from the projector to the screen in feet. At what distance will the projected picture have an area of 100 square feet?

41. **FRAMING** A picture has a square frame that is 2 inches wide. The area of the picture is one third of the total area of the picture and frame. What are the dimensions of the picture to the nearest quarter of an inch?

Solve each equation by using the Square Root Property.

42. $x^2 + x + \frac{1}{4} = \frac{9}{16}$

43. $x^2 + 1.4x + 0.49 = 0.81$

44. $4x^2 - 28x + 49 = 5$

45. $9x^2 + 30x + 25 = 11$

Find the value of c that makes each trinomial a perfect square. Then write the trinomial as a perfect square.

46. $x^2 + 0.6x + c$

47. $x^2 - 2.4x + c$

48. $x^2 - \frac{8}{3}x + c$

49. $x^2 + \frac{5}{2}x + c$

Solve each equation by completing the square.

50. $x^2 + 1.4x = 1.2$

51. $x^2 - 4.7x = -2.8$

52. $x^2 - \frac{2}{3}x - \frac{26}{9} = 0$

53. $x^2 - \frac{3}{2}x - \frac{23}{16} = 0$

54. $3x^2 - 4x = 2$

55. $2x^2 - 7x = -12$

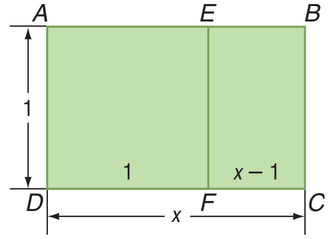
56. ENGINEERING In an engineering test, a rocket sled is propelled into a target. The sled's distance d in meters from the target is given by the formula $d = -1.5t^2 + 120$, where t is the number of seconds after rocket ignition. How many seconds have passed since rocket ignition when the sled is 10 meters from the target?

Real-World Link

Reverse ballistic testing—accelerating a target on a sled to impact a stationary test item at the end of the track—was pioneered at the Sandia National Laboratories' Rocket Sled Track Facility in Albuquerque, New Mexico. This facility provides a 10,000-foot track for testing items at very high speeds.

Source: sandia.gov

GOLDEN RECTANGLE For Exercises 57–59, use the following information.
A *golden rectangle* is one that can be divided into a square and a second rectangle that is geometrically similar to the original rectangle. The ratio of the length of the longer side to the shorter side of a golden rectangle is called the *golden ratio*.

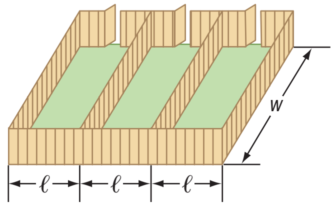


57. Find the ratio of the length of the longer side to the length of the shorter side for rectangle *ABCD* and for rectangle *EBCF*.

58. Find the exact value of the golden ratio by setting the two ratios in Exercise 57 equal and solving for x. (*Hint:* The golden ratio is a positive value.)

59. RESEARCH Use the Internet or other reference to find examples of the golden rectangle in architecture. What applications does the golden ratio have in music?

60. KENNEL A kennel owner has 164 feet of fencing with which to enclose a rectangular region. He wants to subdivide this region into three smaller rectangles of equal length, as shown. If the total area to be enclosed is 576 square feet, find the dimensions of the enclosed region. (*Hint:* Write an expression for ℓ in terms of w.)



EXTRA PRACTICE
See pages 901, 930.

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H.O.T. Problems

61. OPEN ENDED Write a perfect square trinomial equation in which the linear coefficient is negative and the constant term is a fraction. Then solve the equation.

62. FIND THE ERROR Rashid and Tia are solving $2x^2 - 8x + 10 = 0$ by completing the square. Who is correct? Explain your reasoning.

Rashid

$$2x^2 - 8x + 10 = 0$$
$$2x^2 - 8x = -10$$
$$2x^2 - 8x + 16 = -10 + 16$$
$$(x - 4)^2 = 6$$
$$x - 4 = \pm\sqrt{6}$$
$$x = 4 \pm \sqrt{6}$$

Tia

$$2x^2 - 8x + 10 = 0$$
$$x^2 - 4x = 0 - 5$$
$$x^2 - 4x + 4 = -5 + 4$$
$$(x - 2)^2 = -1$$
$$x - 2 = \pm i$$
$$x = 2 \pm i$$

63. REASONING Determine whether the value of c that makes $ax^2 + bx + c$ a perfect square trinomial is *sometimes*, *always*, or *never* negative. Explain your reasoning.

64. **CHALLENGE** Find all values of n such that $x^2 + bx + \left(\frac{b}{2}\right)^2 = n$ has

a. one real root. **b.** two real roots. **c.** two imaginary roots.

65. ***Writing in Math*** Use the information on page 268 to explain how you can find the time it takes an accelerating car to reach the finish line. Include an explanation of why $t^2 + 22t + 121 = 246$ cannot be solved by factoring and a description of the steps you would take to solve the equation.

STANDARDIZED TEST PRACTICE

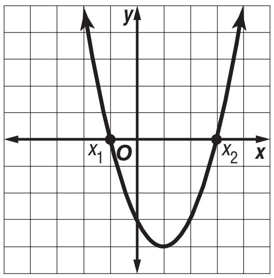
66. **ACT/SAT** The two zeros of a quadratic function are labeled x_1 and x_2 on the graph. Which expression has the greatest value?

A $2x_1$

B x_2

C $x_2 - x_1$

D $x_2 + x_1$



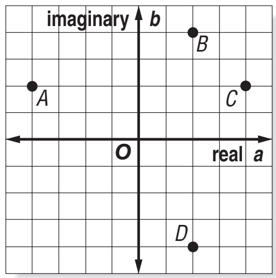
67. **REVIEW** If $i = \sqrt{-1}$ which point shows the location of $2 - 4i$ on the plane?

F point A

G point B

H point C

J point D



Spiral Review

Simplify. (Lesson 5-4)

68. i^{14}

69. $(4 - 3i) - (5 - 6i)$

70. $(7 + 2i)(1 - i)$

Solve each equation by factoring. (Lesson 5-3)

71. $4x^2 + 8x = 0$

72. $x^2 - 5x = 14$

73. $3x^2 + 10 = 17x$

Solve each system of equations by using inverse matrices. (Lesson 4-8)

74. $5x + 3y = -5$
$7x + 5y = -11$

75. $6x + 5y = 8$
$3x - y = 7$

CHEMISTRY **For Exercises 76 and 77, use the following information.**
For hydrogen to be a liquid, its temperature must be within 2°C of −257°C. (Lesson 1-4)

76. Write an equation to determine the least and greatest temperatures for this substance.

77. Solve the equation.

GET READY for the Next Lesson

PREREQUISITE SKILL **Evaluate $b^2 - 4ac$ for the given values of a, b, and c.** (Lesson 1-1)

78. $a = 1, b = 7, c = 3$

79. $a = 1, b = 2, c = 5$

80. $a = 2, b = -9, c = -5$

81. $a = 4, b = -12, c = 9$